

EQUIDISTRIBUTION OF GENERALIZED DEDEKIND SUMS AND EXPONENTIAL SUMS

BYUNGHEUP JUN AND JUNGYUN LEE

ABSTRACT. For the generalized Dedekind sums $s_{ij}(p, q)$ defined in association with the $x^i y^j$ -coefficient of the Todd power series of the lattice cone in \mathbb{R}^2 generated by $(1, 0)$ and (q, p) , we prove the equidistribution of the fractional parts of $R(i, j)q^{i+j-2}s_{ij}(p, q)$ for a certain integer $R(i, j)$ depending on i and j . Using the cocycle condition satisfied by Todd series of 2d cones, we consider certain cases of exponential sums via continued fraction of q/p . Its Weil bound is given for the modulus q as a consequence of the purity of the cohomology of the related $\bar{\mathbb{Q}}_\ell$ -sheaf due to Denef and Loeser. From this, we prove that the Weyl's equidistribution criterion is fulfilled for the fractional part of $R(i, j)q^{i+j-2}s_{ij}(p, q)$. As a special case, we recover the equidistribution result of the classical Dedekind sums multiplied by 12 not using the modular weight of the Dedekind's $\eta(\tau)$.

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1. INTRODUCTION

Classical Dedekind sum $s(p, q)$ is defined for relatively prime integers p, q by

$$s(p, q) = \sum_{k=1}^q \left(\left\{ \frac{k}{p} \right\} \right) \left(\left\{ \frac{kp}{q} \right\} \right)$$

where $\{-\}$ denotes the fractional part. This appears important in describing the change of the Dedekind eta function

$$\eta(\tau) = e^{2\pi i \tau / 24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad \tau \in \mathfrak{h},$$

under modular transformations. $\eta(\tau)$ is a 24th root of the modular discriminant

$$\Delta(\tau) = (12\pi)^{12} \eta^{24}(\tau)$$

up to some constant.

Due to the modularity after 24th power, under modular transformation given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (assuming $c \neq 0$!), $\tau \mapsto A\tau$, its logarithm satisfies the following formula due to Dedekind:

$$(1) \quad \log \eta \left(\frac{a\tau + b}{c\tau + d} \right) = \log \eta(\tau) + \frac{1}{4} \log \left(-(c\tau + d)^2 \right) + \pi i \phi(A)$$

where $\phi(-)$ is the Rademacher's ϕ -function defined as

$$\phi(A) = \frac{a+d}{12c} - \mathrm{sign} c \cdot s(a, c) \in \frac{1}{12}\mathbb{Z}.$$

ϕ is defined by Rademacher $\phi : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \frac{1}{12}\mathbb{Z}$ and many interesting properties of Dedekind sums arise in study of ϕ .

Rademacher and Grosswald raised a question on the density of the values of Dedekind sums in p.28 of [24]. There is a long list of publications on this direction. Hickerson proved the density of the points $(p/q, s(p, q))$ in the plane ([11]). Much later, big progress was made independently by Myerson and Vardi. Vardi showed that for any positive real number r , the set $\{\langle rs(p, q) \rangle \mid q > 0, 0 < p < q, (p, q) = 1\}$ is equidistributed on the interval $[0, 1)$ by relating Kloosterman sums to Dedekind sums ([31]). Similar method is applied by Myerson in [21] to show that the graph of the function $p/q \mapsto rs(p, q)$ is equidistributed (modulo 1).

Let us briefly recall the idea of Vardi and Myerson. As ϕ is valued in $\frac{1}{12}\mathbb{Z}$, for the case $r \in 12\mathbb{Z}$, one can relate the Kloosterman sums to the Dedekind sums multiplied by 12 (Thm.2.1). There is a well-known bound of Kloosterman sums for varying modulus due to Weil, which is crucial step in showing the Weyl's equidistribution criterion for the fractional part of Dedekind sums. For $r \notin 12\mathbb{Z}$, a generalization of Kloosterman sum due to Selberg is associated to the multiplier system arising from Dedekind's η -function ([27]).

This will be reviewed more precisely in Sec.2 of this article for our own purpose.

In this article, we note that many other properties of Dedekind sums are consequence of cocycle condition. As a cochain over $(P)SL_2(\mathbb{Z})$, the coboundary of ϕ is identified with either the area 2-cocycle or the signature 2-cocycle of torus fibration over a pair of pants(cf. [3], [12], [17], [20]). Since these 2-cocycles have simple geometric interpretations, one obtains identities involving Dedekind sums (cf. [3], [12], [17], [26]). It is a folklore that all known properties are consequences of the cocycle condition of this sort(e.g. [29], [28], [26]).

To name one of them, one obtains the celebrated reciprocity law for the Dedekind sums by swapping the two arguments. Also from the (finite length) continued fraction of a rational number one can describe it using the terms of the continued fraction(cf. [4], [11], [18]). Of course, a brute-force computation using the explicit expression recovers the reciprocity law again, but the explicit form itself is already the consequence of the cocycle condition.

The goal of this article is obtaining the equidistribution result for a version of generalization of Dedekind sums appearing in a paper of Apostol([1]) from the cocycle condition. This method specialized to the classical Dedekind sums yields the same sort of equidistribution.

For $i, j \geq 1$, we consider the following generalization of Dedekind sums:

$$s_{ij}(p, q) := \sum_k \bar{B}_i\left(\frac{k}{q}\right) \bar{B}_j\left(\frac{pk}{q}\right).$$

where $\bar{B}_i(x)$ denotes the periodic Bernoulli function. Classical Dedekind sums occur for the case $i = j = 1$. These appeared first in *loc.cit.* and Carlitz wrote some papers on their properties([6], [7]).

There are many ways of writing the Dedekind sums. In this paper, so as to treat the generalized ones as well as the classical, we recover the Dedekind sums as the coefficients of (the germ of) a certain analytic function at 0 in \mathbb{C}^2 associate to a 2-dimensional lattice cone in \mathbb{R}^2 . This idea was taken by Solomon([28]) on a different basis. A lattice cone can be identified with an affine linear map $\sigma : [0, 1] \rightarrow \mathbb{R}^2$ such that $\sigma(0), \sigma(1)$ are two linearly independent primitive lattice vectors. Seen as a singular chain of $\mathbb{R}^2 - 0$, we have obvious notion of the boundary operation. By (1-)cocycle, we mean a functional S defined on 2-dimensional cones, which vanishes on the boundary of a (degenerate) 3-dimensional cone. This is equivalent to say that

$$S(\sigma_1) + S(\sigma_2) = S(\sigma)$$

for $\sigma_1 + \sigma_2 = \sigma$ where the addition of cones is defined as their concatenation.

For a pair of relatively prime positive integers $p, q (p > q)$, one associates a two variable power series denoted by $\text{Todd}_{pq}(x, y)$ (For precise definition, we refer the reader to SS3.2). Todd_{pq} is the Todd series of the cone $\sigma((1, 0), (p, q))$, whose coefficients $\frac{t_{ij}(p, q)}{i!j!}$ of the monomial $x^i y^j$ is closely related to $s_{ij}(p, q)$.

In particular, for nonsingular cones, it is

$$(2) \quad \text{Todd}(x, y) = \frac{x}{1 - e^{-x}} \frac{y}{1 - e^{-y}} = \sum_{i,j} \frac{B_i B_j}{i! j!} x^i y^j$$

Todd series, after certain normalization, makes a cocycle, which we call Todd cocycle in this article. Using the cocycle condition, we obtain an explicit formula of Todd series of a cone w.r.t. the cone decomposition attached to the continued fraction. Since the cones appearing in continued fraction are nonsingular, the Todd series is decomposed into the Todd series of each nonsingular cones. As seen in (19), we finally obtain an expression of generalized Dedekind sums involving only the finite number of Bernoulli numbers. Then multiplied by the denominator which appear to be $R(a, b)q^N$ in Thm.1.1 below, we associate certain exponential sums, which generalizes the occurrence of Kloosterman sums for classical case (See Def.5.4). This generalization of Kloosterman sums have Weil type bound. It is a consequence of a result on the weight and dimension of the cohomology of a ℓ -adic sheaf due to Denef-Loeser in [9]. From the Weil bound for the generalized Kloosterman sums, one can show that the Weyl's equidistribution criterion for the generalized Dedekind sums is achieved.

Our main theorem states the equidistribution property of the (fractional part) of the generalized Dedekind sums similar to Vardi's on classical Dedekind sums in [31].

Theorem 1.1 (main theorem). *For even $N = i + j$, the set*

$$\{\langle R(i, j)q^{N-2}s_{i,j}(p, q) \rangle \mid 0 < p < q, (p, q) = 1\}$$

is equidistributed in the interval $[0, 1)$, where $\langle x \rangle$ is the fractional part of x in $[0, 1)$.

Said roughly, the equidistribution of the classical Dedekind sums multiplied by 12 in *loc.cit.* is a consequence of the modularity of $\eta(\tau)$. Contrary to the classical case, as we don't have such a function which play the role of $\log \eta(\tau)$ for generalized Dedekind sums, the theorem is not entirely clear from the definition. Thus we speculate that this might be a consequence the cocycle property of the Todd series similar to many other properties of the classical Dedekind sums.

Actually, Vardi obtained much stronger result for classical Dedekind sums. He showed that the fractional part of r -multiple of the Dedekind sums are equidistributed for arbitrary nonzero real number r . The multiplier system associates Selberg's generalization of Kloosterman sums to these numbers. Unfortunately, at the moment, we don't have such strong equidistribution for generalized Dedekind sums due to absence of $\log \eta(\tau)$. We expect that certain Lambert series considered by Apostol([1]) instead of $\log \eta(\tau)$ might be applied to this question as in the proof of the generalized Petersson-Knopp identities by Parson-Rosen([22]).

This paper is composed as follows: In Sec. 2, we review Vardi's method to relate the Kloosterman sum to the classical Dedekind sums. In Sec. 3, we define the Todd

series of a lattice cone and describe the cocycle condition. In Sec. 4, the generalized Dedekind sums are identified with the coefficients of the Todd series. In Sec. 5 the generalized Kloosterman sums appear in relation to the generalized Dedekind sums. Sec. 6. is devoted to the Weil bound for generalized Kloosterman sums and we finish the proof of the main theorem.

Notations and convention

- For a real number x , $\langle x \rangle = x - [x]$ is the fractional part taken in $[0, 1)$.
- For a function f in x , $f \ll_{\epsilon} x^a$ means that $f = o(x^{a+\epsilon})$ for every positive ϵ .
- $e(x)$ denotes $\exp(2\pi i x)$.
- The k -th Bernoulli number B_k is defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k.$$

- The k -th Bernoulli polynomial $B_k(x)$ is the degree k polynomial defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k.$$

The k -th periodic Bernoulli function $\bar{B}_k(x)$ is a \mathbb{Z} -periodic function on \mathbb{R} defined by assigning the values for $x \in [0, 1)$ as follows:

$$\bar{B}_k(x) = \begin{cases} B_k(x) & \text{for } x \in (0, 1) \\ B_k & \text{for } k \geq 2 \text{ and } x = 0, \\ 0 & \text{for } k = 1, x = 0 \end{cases}$$

2. EQUIDISTRIBUTION OF CLASSICAL DEDEKIND SUMS

In this section, we sketch the proof of the equidistribution of the fractional parts of classical Dedekind sums multiplied by 12 using the Weil's bound for Kloosterman sums. The proof appears at the beginning of Vardi([31]). This is not only a special case of the main result of *loc.cit.* but also a case not covered by the main technic of multiplier system attached to $\log \eta(\tau)$.

As discussed, this step comes from the modularity of $\eta(\tau)$ in relation to $\Delta(\tau)$.

We remind a criterion for a sequence to be equidistributed in $[0, 1)$ due to H. Weyl and will show that this is the case. Later, we will be using the Weyl's equidistribution criterion for the generalized Dedekind sums.

2.1. Rademacher's theorem. We begin with a reinterpretation of Rademacher's ϕ -function.

Theorem 2.1 (Rademacher). *For a relatively prime pair of integers (p, q) , $12s(p, q) - \frac{p'+p}{q}$ is always integer whenever $p'p \equiv 1 \pmod{q}$.*

The above Rademacher's theorem is nothing but rewriting the fact that the values of Rademacher's ϕ -function are taken in $\frac{1}{12}\mathbb{Z}$. At a glance, this is not along the line we follow to show the result of same type for generalized Dedekind sums. But later we will see that the integrality will turn out to be a special case of the cocycle property. Later in Sec. 5, we will see that the appearance of 12 in the Rademacher's theorem is due to the denominators are product of B_1 and B_2 the 1st and 2nd Bernoulli numbers.

2.2. Kloosterman sums. Let q be a positive integer and k, ℓ be a pair of integers relatively prime to q . The *Kloosterman sum* for k, ℓ of modulus m is denoted by $K(k, \ell, q)$ and defined as

$$K(k, \ell, q) := \sum_{x \in (\mathbb{Z}/m\mathbb{Z})^*} \mathfrak{e}\left(\frac{k}{q}x + \frac{\ell}{q}x^{-1}\right) = \sum_{\substack{0 \leq x \leq m-1 \\ (x, m)=1}} \mathfrak{e}\left(\frac{k}{q}x + \frac{\ell}{q}x^{-1}\right)$$

There is a well-known upper bound of Kloosterman sums due to Weil:

$$(3) \quad K(k, \ell, q) = o(q^{\frac{1}{2}+\epsilon}), \quad (\forall \epsilon > 0).$$

2.3. Kloosterman sum and Dedekind sum. For two positive integers m and q fixed, if we sum $e(12ms(p, q))$ over $1 \leq p \leq q$ such that $(p, q) = 1$, we obtain the Kloosterman sum from Thm.2.1:

$$(4) \quad K(m, m, q) = \sum_{\substack{0 < p < q \\ (p, q)=1}} \mathfrak{e}(12ms(p, q)) = \sum_{\substack{0 < p < q \\ (p, q)=1 \\ p'p \equiv 1 \pmod{q}}} \mathfrak{e}\left(\frac{p'm + pm}{q}\right).$$

Using Weil's bound (3), we conclude that

$$(5) \quad \sum_{0 < q < x} \sum_{\substack{0 < p < q \\ (p, q)=1}} \mathfrak{e}(12ms(p, q)) = o(x^{\frac{3}{2}+\epsilon}), \quad (\forall \epsilon > 0).$$

Then the above bound implies that the set

$$\{\langle 12s(p, q) \rangle \mid q > 0, 0 < p < q, (q, p) = 1\}$$

fulfills a famous criterion for a sequence in $[0, 1)$ to be equidistributed due to H. Weyl, where $\langle x \rangle \in [0, 1)$ denotes the fractional part of x .

2.4. Weyl's equidistribution criterion. A sequence $\{s_i \in [0, 1)\}_{i \in \mathbb{N}}$ is equidistributed iff for all $k \in \mathbb{Z}$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathfrak{e}(ks_n) = 0.$$

For a positive integer m and x , let $E(m, x)$ be the value

$$E(m, x) = \frac{1}{\#\{(p, q) \mid \gcd(p, q) = 1, p < q \leq x\}} \sum_{0 < q < x} \sum_{0 < p < q, (p, q)=1} \mathfrak{e}(12ms(p, q)).$$

Taking the limit $x \rightarrow \infty$, from the Weil's bound, we have $E(m, x) \rightarrow 0$. Therefore $\langle 12s(p, q) \rangle$ is equidistributed in $[0, 1)$.

For the rest of the paper, we consider the possibility of the same sort of equidistribution of the generalized Dedekind sums $s_{ij}(p, q)$.

3. TODD SERIES OF A LATTICE CONE

Now we recall the definition of Todd series defined in [5], for the case of 2-dimensional cones. For 1 dimensional case, this equals the generating function of the Bernoulli numbers up to change of the sign of the variable. The Todd series yields a differential operator of infinite order used in the formulation of the Euler-Maclaurin formula for higher dimensional polytopes as well as for one dimension. These generating functions are closely related to Shintani functions considered by Solomon in [28] and have similar cocycle property. Later we will see that the Todd series recovers the generalized Dedekind sums as Solomon's Shintani functions. But we warn the reader that these are not the same *a priori* their cocycle conditions are on the dual cones to each other. This is briefly mentioned by Garoufalidis-Pommersheim in [10]. We will clarify this relation in another paper in sequel [15]. We may conclude the same result in the framework of Shintani functions. But through our works in relation([14], [15]), we find it more comfortable to manipulate Todd series rather than Shintani functions.

3.1. Lattice cones. Let M be the lattice \mathbb{Z}^2 in \mathbb{R}^2 . We consider cones defined in M . By lattice cone, we mean the convex hull of two linearly independent rays of rational slopes. It is always possible to choose unique primitive lattice vectors generating the rays. Let $\sigma = \sigma(v_1, v_2)$ be a lattice cone and v_1, v_2 be primitive lattice generators of the rays bounding σ . σ is sometimes identified with an integer coefficient matrix A_σ whose columns are the lattice vectors v_1, v_2 in \mathbb{Z}^2 . M_σ denotes the sublattice of M generated by v_1, v_2 . $\Gamma_\sigma = M/M_\sigma$ is isomorphic to a cyclic group of order $|\det(A_\sigma)|$.

For $g \in M$ representing $\gamma \in \Gamma_\sigma$, we have rational numbers $a_{\sigma,i}(g)$, $i = 1, 2$ such that

$$g = a_{\sigma,1}(\gamma)v_1 + a_{\sigma,2}(\gamma)v_2.$$

$a_{i,\sigma}$, being integral on M_σ , yields a character $\chi_{\sigma,i}$ on Γ_σ as

$$\chi_{\sigma,i}(\gamma) := \mathfrak{e}(a_{\sigma,i}(g)), \quad \text{for } i = 1, 2.$$

3.2. Todd Series. The Todd power series of σ is defined as:

$$(6) \quad \text{Todd}_\sigma(x_1, x_2) := \sum_{\gamma \in \Gamma_\sigma} \frac{x_1}{1 - \chi_{\sigma,1}(\gamma)e^{-x_1}} \frac{x_2}{1 - \chi_{\sigma,2}(\gamma)e^{-x_2}}$$

The coefficients of $\text{Todd}_\sigma(x_1, x_2)$ is rational though the expression (6) contains some roots of 1. This is easy to see from the Galois invariance of the expression.

The Todd series is invariant of the $\text{SL}_2(\mathbb{Z})$ equivalent class of cones by the following proposition.

Proposition 3.1. *Let $\sigma = \sigma(v_1, v_2)$ be a rational simple cone and A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$. Then we have*

$$\mathrm{Todd}_\sigma(x_1, x_2) = \mathrm{Todd}_{A\sigma}(x_1, x_2).$$

Proof. A gives an isomorphism

$$A : M/M_\sigma \rightarrow M/M_{A\sigma}, \quad \gamma \mapsto A\gamma, \text{ for } \gamma \in \Gamma_\sigma.$$

For $\gamma = a_{\sigma,1}(\gamma)v_1 + a_{\sigma,2}(\gamma)v_2$,

$$A\gamma = a_{\sigma,1}(\gamma)Av_1 + a_{\sigma,2}(\gamma)Av_2.$$

Therefore

$$\begin{aligned} \mathrm{Todd}_{A\sigma}(x_1, x_2) &= \sum_{A\gamma \in \Gamma_{A\sigma}} \frac{x_1}{1 - \chi_{A\sigma,1}(A\gamma)e^{-x_1}} \frac{x_2}{1 - \chi_{A\sigma,2}(A\gamma)e^{-x_2}} \\ &= \sum_{A\gamma \in \Gamma_{A\sigma}} \frac{x_1}{1 - \chi_{\sigma,1}(\gamma)e^{-x_1}} \frac{x_2}{1 - \chi_{\sigma,2}(\gamma)e^{-x_2}} \\ &= \mathrm{Todd}_\sigma(x_1, x_2) \end{aligned}$$

□

Let $p, q > 0$ be two relatively prime nonnegative integers. Then $(1, 0)$ and (p, q) are primitive lattice vectors and linearly independent. Let $\sigma_{p,q}$ denote the cone generated by $(1, 0)$ and (p, q) . Notice that any lattice cone is equivalent to $\sigma_{p,q}$ after basis change. We shall write Todd_{pq} instead of $\mathrm{Todd}_{\sigma_{p,q}}$ for simplicity.

3.3. Normalized Todd series and cocycle property. From now on, we will be dealing with only lattice cones in the 1st quadrant. This is not necessary in defining the cocycle property for cones but otherwise we need to extend the category of cones due to the occurrence of the Maslov index(cf. [2], [15], [28]). In [28], this ambiguity appeared by the name ‘*formal Cauchy theorem*’. Consequently, for full generality, one has to take the value of the cocycle modulo \mathbb{Z} (eg.[28]) or to take a central extension of $\mathrm{SL}_2(\mathbb{Z})$ (eg.[3]). Here considering only the cones in the 1st quadrant, we can avoid this difficulty. A drawback is that we don’t have the cocycles defined over $\mathrm{GL}_2(\mathbb{Q})$ but over singular chains in $\mathbb{R}^2 - 0$, but this is enough for this article.

Definition 3.2. *Let σ be a lattice cone. Then the normalized Todd series $S_\sigma(x_1, x_2)$ of σ is defined as*

$$S_\sigma(x_1, x_2) = \frac{1}{\det(A_\sigma)x_1x_2} \mathrm{Todd}_\sigma(x_1, x_2).$$

Similarly, $S_{\sigma_{p,q}}$ is abbreviated to S_{pq} as in unnormalized case.

Because $\mathrm{Todd}_\sigma(x_1, x_2)$ is holomorphic at $0 \in \mathbb{C}^2$, we may well take $\mathrm{Todd}_\sigma(x_1, x_2) \in \mathbb{C}\{\{x_1, x_2\}\}$, where $\mathbb{C}\{\{x_1, x_2\}\}$ is the ring of power series convergent for some

neighborhood of 0. Note that the coefficients of $\text{Todd}_\sigma(x_1, x_2)$ lie in \mathbb{Q} . Thus we can write

$$\text{Todd}_\sigma(x_1, x_2) \in \mathbb{Q}\{\{x_1, x_2\}\} := \mathbb{C}\{\{x_1, x_2\}\} \cap \mathbb{Q}[[x_1, x_2]].$$

With the notations above, since $S_\sigma(x_1, x_2)$ has simple pole along the two axes: $x_1 = 0$ and $x_2 = 0$,

$$S_\sigma(x_1, x_2) \in \frac{1}{x_1 x_2} \mathbb{Q}\{\{x_1, x_2\}\}.$$

Note that swapping two rays of the cone interchanges not only the variables but also the sign in S_σ . Thus the orientation of a cone is reflected in S_σ . In this case, the same cone with the opposite orientation will be denoted by $-\sigma$.

Lemma 3.3. *For a lattice cone σ in the 1st quadrant,*

$$\begin{aligned} S_{-\sigma}(x_1, x_2) &= -S_\sigma(x_2, x_1) \\ \text{Todd}_{-\sigma}(x_1, x_2) &= \text{Todd}_\sigma(x_2, x_1) \end{aligned}$$

Let v_1, v_2, v_3 be pairwise linearly independent primitive lattice vectors in the 1st quadrant. Let σ_{ij} be the lattice cone generated by v_i, v_j . Then we write $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$. Actually a cone σ generated by lattice vectors v_1, v_2 can be seen as a simplex

$$\sigma : [0, 1] \rightarrow \mathbb{R}^2$$

for an affine linear map σ with $\sigma(0) = v_1, \sigma(1) = v_2$. Abusing the notation, if $v_1 = v_2$, such a degenerate cone behaves as a unit when added to $\sigma(v_1, v)$ for any v . Thus we consider the groupoid of 2 dimensional cones. The addition is defined up to boundary of (degenerate) 3 dimensional cones.

Definition 3.4. *A 1-cocycle over cones is a functional over cones valued in an abelian group M*

$$\phi : \sigma \mapsto \phi(\sigma) \in M$$

satisfying $\phi(\sigma_1 + \sigma_2) = \phi(\sigma_1) + \phi(\sigma_2)$.

In terms of groupoid, the 1-cocycle is a map preserving the operation of the groupoid of cones.

To avoid unnecessary complication, we will consider only the cones lying in the right half plane $\mathbb{R}_{x_1 \geq 0}^2$. From now on, we denote by ‘2-d Cones’ the set of all 2-dimensional lattice cones in the right half plane. Todd cocycle is a cocycle on 2-d Cones defined as below. Let L be the reduced equation of the line orthogonal to a lattice vector (p, q) for $p > 0$. Written explicitly,

$$L = qx_1 - px_2.$$

By $\frac{1}{\prod_L} \mathbb{Q}\{\{x_1, x_2\}\}$, we mean

$$\sum_L \frac{1}{L} \mathbb{Q}\{\{x_1, x_2\}\}$$

where L runs for all primitive lattice vectors in the right half plane.

Definition-Proposition 3.5 (Todd cocycle). *The Todd cocycle is a map*

$$\Phi : 2\text{-d Cones} \rightarrow \frac{1}{\prod_L L} \mathbb{Q}\{\{x_1, x_2\}\}$$

given by

$$\Phi(\sigma) = S_\sigma(A_\sigma^{-1}(x_1, x_2)) \in \mathbb{Q}\{\{x_1, x_2\}\}(L_1^{-1}, L_2^{-1}),$$

where L_i are the equation of lines orthogonal to v_1, v_2 , is a 1-cocycle.

Proof. For the proof, we refer the reader to Thm 3. of [23]. □

4. GENERALIZED DEDEKIND SUM AS COEFFICIENTS OF TODD SERIES OF A CONE

Now we are going to identify the generalized Dedekind sums $s_{ij}(p, q)$ with the coefficient of Todd_{pq} . We begin with the definition of $s_{ij}(p, q)$.

Definition 4.1. *For positive integers i, j , define the generalized Dedekind sum as follows:*

$$s_{ij}(p, q) := \sum_{k=0}^{q-1} \bar{B}_i\left(\frac{k}{q}\right) \bar{B}_j\left(\frac{pk}{q}\right).$$

For $i = j = 1$, we have the classical Dedekind sum: $s_{11}(p, q) = s(p, q)$.

Let $\frac{t_{ij}(p, q)}{i!j!}$ be the coefficient of $x_1^i x_2^j$ in $\text{Todd}_{pq}(x_1, x_2)$. Thus

$$\text{Todd}_{pq}(x_1, x_2) = \sum_{i, j \geq 0} \frac{t_{ij}(p, q)}{i!j!} x_1^i x_2^j.$$

Theorem 4.2. *Then, we have*

$$t_{ij}(p, q) = (-q)^{i+j-1} \left(s_{ij}(p, q) + (-1)^{i+j} \delta(i, j) B_i B_j \right),$$

where $\delta(i, j) = \begin{cases} 1, & i = 1 \text{ or } j = 1, \\ 0, & \text{otherwise} \end{cases}$ and B_i is the i -th Bernoulli number.

Proof. Let $\sigma = \sigma_{pq}$ and $v_1 = (1, 0)$, $v_2 = (p, q)$ be the primitive nonzero lattice generators of σ . Let v_j^* be the dual vector of v_i for $i = 1, 2$ (ie. $\langle v_i, v_j^* \rangle = \delta_{ij}$). Thus

$$v_1^* = \left(1, -\frac{p}{q} \right) \quad \text{and} \quad v_2^* = \left(0, \frac{1}{q} \right).$$

The dual cone $\check{\sigma}$ of σ is generated by v_1^* and v_2^* .

Then we have

$$\begin{aligned} \text{Todd}_{pq}(x_1, x_2) &= \sum_{g \in \Gamma_{\sigma_{pq}}} \frac{x_1}{1 - e^{2\pi i \langle v_1^*, g \rangle}} \frac{x_2}{1 - e^{2\pi i \langle v_2^*, g \rangle}} e^{-x_1 - x_2} \\ (7) \quad &= x_1 x_2 \sum_{n_1, n_2 \geq 0} \sum_{g \in \Gamma_{\sigma_{pq}}} e^{2\pi i \langle n_1 v_1^* + n_2 v_2^*, g \rangle} e^{-n_1 x_1 - n_2 x_2}. \end{aligned}$$

Since

$$\sum_{g \in \Gamma_{\sigma_{p,q}}} e^{2\pi i \langle n_1 v_1^* + n_2 v_2^*, g \rangle} = \begin{cases} |\Gamma_{\sigma_{p,q}}| = q, & n_1 v_1^* + n_2 v_2^* \in M^*, \\ 0, & \text{otherwise} \end{cases}$$

one can write Todd_{pq} as summation over lattice points in $\check{\sigma}$:

$$\begin{aligned} \text{Todd}_{pq}(x_1, x_2) &= qx_1 x_2 \sum_{\substack{n_1 v_1^* + n_2 v_2^* \in M^* \\ n_1, n_2 \geq 0}} e^{-n_1 x_1 - n_2 x_2} \\ (8) \quad &= qx_1 x_2 \sum_{m \in \mathbb{Z}^2 \cap \check{\sigma}} e^{-\langle m, v_1 \rangle x_1 - \langle m, v_2 \rangle x_2}. \end{aligned}$$

Notice that in general v_i^* are not lattice vectors but the primitive lattice generators of $\check{\sigma}$ are

$$u_1 = (q, p), \quad u_2 = (0, 1).$$

Let $P(u_1, u_2)$ the following half open parallelogram:

$$P(u_1, u_2) = \{x_1 u_1 + x_2 u_2 \mid 0 \leq x_i < 1\}.$$

Then

$$\check{\sigma} \cap M^* = \{z + n_1 u_1 + n_2 u_2 \mid z \in P(u_1, u_2) \cap M^*, n_i \geq 0\}.$$

Thus we have

$$\begin{aligned} \text{Todd}_{pq}(x_1, x_2) &= qx_1 x_2 \sum_{z \in P(u_1, u_2) \cap M^*} \frac{e^{-\langle z, v_1 \rangle x_1 - \langle z, v_2 \rangle x_2}}{(1 - e^{-qx_1})(1 - e^{-qx_2})} \\ (9) \quad &= q^{-1} \sum_{i, j \geq 0} \sum_{z \in P(u_1, u_2) \cap M^*} B_i \left(\frac{\langle z, v_1 \rangle}{q} \right) B_j \left(\frac{\langle z, v_2 \rangle}{q} \right) (-q)^{i+j} \frac{x_1^i x_2^j}{i! j!} \end{aligned}$$

Since the lattice points inside $P(u_1, u_2)$ are identified as follows:

$$P(u_1, u_2) \cap M^* = \left\{ \frac{k}{q} u_1 + \left(\frac{pk}{q} - \left\lfloor \frac{pk}{q} \right\rfloor \right) u_2 \mid k = 0, 1, 2, \dots, q-1 \right\}.$$

Hence $t_{ij}(p, q)$ and $s_{ij}(p, q)$ are related in the desired form:

$$\begin{aligned} t_{ij}(p, q) &= q^{-1} (-q)^{i+j} \sum_{z \in P(u_1, u_2) \cap M^*} B_i \left(\frac{\langle z, v_1 \rangle}{q} \right) B_j \left(\frac{\langle z, v_2 \rangle}{q} \right) \\ &= q^{-1} (-q)^{i+j} \sum_{k=0}^{q-1} B_i \left(\frac{k}{q} \right) B_j \left(\left\langle \frac{pk}{q} \right\rangle \right) \\ &= q^{-1} (-q)^{i+j} \left(\sum_{k=0}^{q-1} \bar{B}_i \left(\frac{k}{q} \right) \bar{B}_j \left(\frac{pk}{q} \right) - \delta(i, j) B_i B_j \right) \\ &= -(-q)^{i+j} (s_{ij}(p, q) - \delta(i, j) B_i B_j). \end{aligned}$$

□

For odd $i + j$, $s_{ij}(p, q)$ turns out to be trivial. This is easy consequence of the previous theorem.

Let us define $L^\lambda(x)$ for a fixed complex number $\lambda \neq 0$ as

$$L^\lambda(x) := \frac{x}{2} \frac{1 + \lambda e^{-x}}{1 - \lambda e^{-x}}.$$

For $\lambda = 1$, this is the even part of $\text{Todd}(x)$:

$$\text{Todd}(x) = \frac{x}{2} + L^{\lambda=1}(x)$$

For $\lambda \neq 1$, $L^\lambda(x)$ is not even in general, but we have

$$(10) \quad \text{Todd}^\lambda(x) = \frac{x}{2} + L^\lambda(x)$$

$$(11) \quad L^\lambda(-x) = L^{\lambda^{-1}}(x)$$

Thus if σ is a lattice cone, $\sum_{g \in \Gamma_\sigma} L^{\chi_i(g)}(x_i)$ is even for $i = 1, 2$. Therefore we have decomposition of $\text{Todd}_\sigma(x_1, x_2)$ as follows:

$$(12) \quad \text{Todd}_\sigma(x_1, x_2) = \frac{q}{4} x_1 x_2 + \frac{1}{4} \sum_{g \in \Gamma_\sigma} \left(x_1 L^{\chi_2(g)}(x_2) + x_2 L^{\chi_1(g)}(x_1) \right) + \frac{1}{4} \sum_{g \in \Gamma_\sigma} L^{\chi_1(g)}(x_1) L^{\chi_2(g)}(x_2)$$

Notice that the odd part of $\text{Todd}_\sigma(x_1, x_2)$ is

$$\frac{1}{4} \sum_{g \in \Gamma_\sigma} \left(x_1 L^{\chi_2(g)}(x_2) + x_2 L^{\chi_1(g)}(x_1) \right).$$

So $t_{ij}(p, q) = 0$ for $i + j$ odd and $i, j > 1$. Otherwise, for example $i = 1$ and $j = 2k$ even,

$$t_{1,2k}(p, q) = -\frac{B_{2k}}{2} = s_{1,2k}(p, q) + B_1 B_{2k}.$$

Since $B_1 = -\frac{1}{2}$, again we have $s_{1,2k}(p, q) = 0$.

Hence we obtain the following corollary:

Corollary 4.3. *Let p, q be relatively prime pair of integers. Then for given $i, j \geq 1$ such that $i + j$ is odd,*

$$s_{ij}(p, q) = 0.$$

5. GENERALIZED DEDEKIND SUMS AND GENERALIZED KLOOSTERMAN SUMS

In this section, our goal is to obtain the evaluation of $s_{ij}(p, q)$ in terms of the the continued fraction of q/p for even $i + j = N$. As we saw in the previous section, $s_{ij}(p, q)$ vanishes if $i + j$ is odd.

Writing explicitly $s_{ij}(p, q)$, we will obtain an analogous statement to Rademacher's theorem 2.1. Then we will be able to relate generalized Dedekind sums to a generalization of Kloosterman sums.

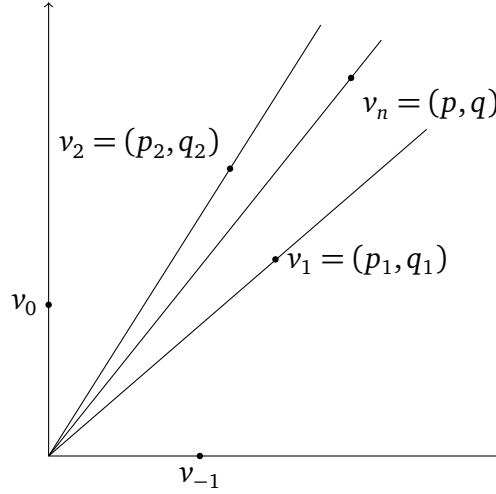


FIGURE 1. asd

The geometric counterpart of the continued fraction is the cone decomposition. Accordingly, the (normalized) Todd series is decomposed into sum of the (normalized) Todd series of nonsingular cones.

Let q and p be relatively prime positive integers and suppose $q > p$.

We are going to associate the (positive) continued fraction of q/p :

$$\frac{q}{p} = a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_n}},$$

where $a_i \geq 1$ are all integers. We put $(p_{-1}, q_{-1}) = (1, 0)$, $(p_0, q_0) = (0, 1)$ and for $i \geq 1$. Define a pair of relatively prime integers p_i and q_i using truncation of the continued fraction of $\frac{q}{p}$:

$$\frac{q_i}{p_i} := a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_i}}.$$

As previous, let $\sigma := \sigma_{p,q}$ and v_k be the primitive lattice vectors in the 1st quadrant (p_k, q_k) for $-1 \leq k \leq n$.

Then we have the following virtual cone decomposition of σ into nonsingular cones:

$$\sigma := \sigma_{pq} = \sigma(v_{-1}, v_n) = \sum_{k=-1}^{n-1} \sigma_k,$$

where $\sigma_k := \sigma(v_k, v_{k+1})$.

After the additivity of normalized Todd series, according to the continued fraction of p/q , we obtain the following expression:

$$(13) \quad S_{pq}(x, y) = \sum_{k=-1}^{n-1} (-1)^{k+1} F \left(A_{\sigma_k}^{-1} A_{\sigma}(x, y)^t \right),$$

where

$$F(x, y) = \frac{1}{1 - e^{-x}} \frac{1}{1 - e^{-y}} = \frac{\text{Todd}_{01}(x, y)}{xy}.$$

One should note that $F(x, y)$ is the normalized Todd of a nonsingular cone (up to sign).

Recall that the 1-variable Todd series is

$$\text{Todd}(z) = \frac{z}{1 - e^{-z}} = \sum_{i=0}^{\infty} (-1)^i \frac{B_i}{i!} z^i.$$

The matrix $A_{\sigma_k}^{-1} A_{\sigma}$ is computed as

$$A_{\sigma_k}^{-1} A_{\sigma} = (-1)^{k+1} \begin{pmatrix} q_{k+1} & pq_{k+1} - qp_{k+1} \\ -q_k & -pq_k + qp_k \end{pmatrix}.$$

As $\det(A_{\sigma}) = q$, by multiplying qxy we obtain the following expression of Todd series of σ from (13):

$$(14) \quad \begin{aligned} \text{Todd}_{pq}(x, y) &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{\text{Todd}(M_k) \text{Todd}(M_{k+1})}{M_k M_{k+1}} \\ &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \frac{B_i}{i!} \frac{B_j}{j!} M_k^{j-1} M_{k+1}^{i-1}, \end{aligned}$$

where

$$(15) \quad M_k := \begin{cases} qy, & k = -1 \\ (-1)^k (q_k x + (pq_k - qp_k)y), & 0 \leq k \leq n-1 \\ (-1)^n qx, & k = n. \end{cases}$$

Denote by Todd_{σ}^N the degree N homogeneous part of Todd_{σ} . Then from (14) Todd_{σ}^N is given as follows:

$$(16) \quad \begin{aligned} \text{Todd}_{\sigma}^N &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} (-1)^N \frac{B_{i+1}}{(i+1)!} \frac{B_{N-i-1}}{(N-i-1)!} M_k^{N-2-i} M_{k+1}^i \\ &\quad + (-1)^N qxy \frac{B_N}{N!} \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{M_k^N + M_{k+1}^N}{M_k M_{k+1}}. \end{aligned}$$

Since M_k satisfies the recursive relation for $k \geq 0$,

$$M_{k-1} - M_{k+1} = a_{k+1}M_k,$$

we have

$$(17) \quad \begin{aligned} & qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \frac{M_k^N + M_{k+1}^N}{M_k M_{k+1}} \\ &= qxy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} M_{k-1}^{N-2-i} M_{k+1}^i \right) + M_0^{N-1}x + M_{n-1}^{N-1}y. \end{aligned}$$

Therefore, plugging (17) into (16), we obtain

$$(18) \quad \begin{aligned} \text{Todd}_{pq}^N &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} (-1)^N \frac{B_{i+1}}{(i+1)!} \frac{B_{N-i-1}}{(N-i-1)!} M_k^{N-2-i} M_{k+1}^i \\ &+ (-1)^N q \frac{B_N}{N!} xy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} M_{k-1}^{N-i-2} M_{k+1}^i \right) + \frac{B_N}{N!} M_0^{N-1}x + \frac{B_N}{N!} M_{n-1}^{N-1}y. \end{aligned}$$

Consequently, we obtain the following integrality involving Todd_{pq}^N .

Theorem 5.1. Let $B_{k+2} = \frac{\alpha_k}{\beta_k}$ be a reduced fraction with $\beta_k \leq 0$ and

$$r_N := \text{L.C.M.} \left\{ \text{Denominator of } \beta_N \binom{N}{i+1} B_{i+1} B_{N-i-1}, 0 \leq i \leq \frac{N-2}{2} \right\}.$$

Then we have

$$\begin{aligned} & N! \beta_N r_N \text{Todd}_{pq}^N(x, y) \\ & - \alpha_N r_N (x + py)^{N-1} x - \alpha_N r_N \left((-1)^{n-1} q_{n-1} x + y \right)^{N-1} y \in q\mathbb{Z}[x, y]. \end{aligned}$$

Proof. By multiplying $N! \beta_N$ on the equation (18), we have that

$$(19) \quad \begin{aligned} & N! \beta_N \text{Todd}_{pq}^N - \alpha_N M_0^{N-1}x - \alpha_N M_{n-1}^{N-1}y \\ &= qxy \sum_{k=-1}^{n-1} (-1)^{k+1} \sum_{i=0}^{N-2} (-1)^N \beta_N \binom{N}{i+1} B_{i+1} B_{N-i-1} M_k^{N-i-2} M_{k+1}^i \\ &+ (-1)^N q \alpha_N xy \left(\sum_{k=0}^{n-1} (-1)^k a_{k+1} \sum_{i=0}^{N-2} M_{k-1}^{N-2-i} M_{k+1}^i \right). \end{aligned}$$

Since $M_i \in \mathbb{Z}[x, y]$ for every i , multiplying (19) by r_N , we conclude the proof. \square

If we read the coefficient of a monomial $x^{i+1}y^{N-i-1}$ in the previous theorem, we have an analogous statement to the Rademacher's theorem.

Theorem 5.2. *With the same assumption and notation, for $i + j = N \geq 2$, we have*

$$R(i, j)q^{N-2}s_{i,j}(p, q) - \frac{(p')^i \alpha_N r_N \binom{N-1}{i} + p^j \alpha_N r_N \binom{N-1}{j}}{q}$$

is always integer whenever $p'p \equiv 1 \pmod{q}$, where

$$R(i, j) := \binom{N}{i} \beta_N r_N$$

Remark 5.3. *In Cor. 5.2, if we consider the case of $i = 1 = j$ and $N = i + j = 2$, then the denominator β_0 of B_2 is 6 and r_N is 1. Thus, we have $R(i, j) = 12$. We note that this is the case of Thm. 2.1.*

From Cor. 5.2, we have

$$(20) \quad \sum_{\substack{0 < p < q \\ (p, q) = 1}} e \left(R(i, j)q^{N-2}s_{i,j}(p, q) \right) = \sum_{\substack{0 < p < q \\ (p, q) = 1 \\ pp' \equiv 1 \pmod{q}}} e \left(\frac{(p')^i \alpha_N r_N \binom{N-1}{i} + p^j \alpha_N r_N \binom{N-1}{j}}{q} \right)$$

Now we define generalized Kloosterman sum as follows:

Definition 5.4 (Generalized Kloosterman sum). *For a positive integer q ,*

$$K_{i,j}(k, \ell, q) := \sum_{\substack{0 < p < q \\ (p, q) = 1 \\ pp' \equiv 1 \pmod{q}}} e \left(\frac{k(p')^i + \ell p^j}{q} \right).$$

Then (20) is a particular case of the generalized Kloosterman sum for $k = \alpha_N r_N \binom{N-1}{i}$, $\ell = \alpha_N r_N \binom{N-1}{j}$.

$$(21) \quad \sum_{\substack{0 < p < q \\ (p, q) = 1}} e \left(R(i, j)q^{N-2}s_{i,j}(p, q) \right) = K_{i,j} \left(\alpha_N r_N \binom{N-1}{i}, \alpha_N r_N \binom{N-1}{j}, q \right).$$

6. BOUNDS FOR GENERALIZED KLOOSTERMAN SUMS

In this section, we are going to investigate the Weil type bound for the generalized Kloosterman sums $K_{ij}(k, \ell, q)$. This estimate amounts basically to asking the weight of the cohomology of a certain ℓ -adic sheaf. From the bound we will show the Weyl's equidistribution criterion for $\{\langle s_{ij}(p, q) \rangle\}$ is fulfilled for $i + j$ even. This will conclude the main theorem.

6.1. Weight of ℓ -adic sheaf and exponential sums. Let us first recall the work of Denef-Loeser([9]). Let X be a scheme of finite type over $k := \mathbb{F}_q$ and $\psi : k \rightarrow \mathbb{C}^*$ be a nontrivial additive character. Then a $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{L}_ψ on \mathbb{A}_k^1 is associated to ψ and the Artin-Schreier covering $t^q - t = x$. For a morphism $f : X \rightarrow \mathbb{A}_k^1$, the exponential sum

$$S(f) = \sum_{x \in X(k)} \psi(f(x))$$

is defined. Let Fr denote the (geometric) Frobenius action. Grothendieck's trace formula identifies the exponential sum with the trace of the Frobenius action:

$$S(f) = \sum_i (-1)^i \text{Tr}(Fr^* | H_c^i(X \otimes \bar{k}, f^* \mathcal{L}_\psi)).$$

For $X = T_k^n$, a map $f : X \rightarrow \mathbb{A}_k^1$ given by a Laurent polynomial $f = \sum_{i \in \mathbb{Z}^n} c_i x^i$, the Newton polyhedron $\Delta_\infty(f)$ is defined as the convex hull of $\{i \in \mathbb{Z}^n | c_i \neq 0\}$ in \mathbb{R}^n . f is said to be *non degenerate w.r.t. $\Delta_\infty(f)$* if for every face σ of $\Delta_\infty(f)$ that does not contain 0, the locus

$$\frac{\partial f_\sigma}{\partial x_1} = \dots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

is empty. Then a result of Denef and Loeser(Thm.1.3. in [9]) is stated as follows:

Theorem 6.1 (Denef-Loeser[9]). *Suppose $f : T_k^n \rightarrow \mathbb{A}_k^1$ is nondegenerate w.r.t. $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = n$. Then we have*

- (1) $H_c^i(T_k^n(\bar{k}), f^* \mathcal{L}_\psi) = 0$ for $i \neq n$,
- (2) $\dim H_c^n(T_k^n(\bar{k}), f^* \mathcal{L}_\psi) = n! \text{Vol}(\Delta_\infty(f))$.
If moreover the interior of $\Delta_\infty(f)$ contains 0, then
- (3) $H_c^n(T_k^n(\bar{k}), f^* \mathcal{L}_\psi)$ is pure of weight n (ie. all Frobenius eigenvalues have absolute value $q^{n/2}$).

If f satisfies the conditions of the above theorem, the trace formula is simplified as

$$S(f) = (-1)^n \text{Tr}(Fr^* | H_c^n(X \otimes \bar{k}, f^* \mathcal{L}_\psi)).$$

Then the Weil type bound is a simple consequence of the purity result:

$$|S(f)| \leq \sum |\text{Frobenius eigenvalue}| \leq C_f q^{n/2},$$

where $C_f = \dim H_c^n(T_k^n(\bar{k}), f^* \mathcal{L}_\psi) = n! \text{Vol}(\Delta_\infty(f))$.

By the fundamental result of Deligne in [8], we know that $H_c^n(T_k^n, f^* \mathcal{L}_\psi)$ has mixed weight $\leq n$. This is already enough to obtain the Weil bound, but to obtain the dimension, we need the theorem of Denef-Loeser.

First, for $q = p$ and $f(z) = kz^i + \ell z^{-j}$ non degenerate, we obtain the Weil bound for the generalized Kloosterman sum. Second, we reduce the generalized Kloosterman sum of composite modulus to a product of those of p -primary modulus. Besides, we will consider those exceptional cases separately. Altogether, the bound will yield the Weyl's criterion for equidistribution.

6.2. Reduction to non degenerate case. Let us write first the Weil bound for the generalized Kloosterman sum of prime modulus in non degenerate case.

Lemma 6.2 (Nondegenerate case). *Let p be a prime and p does not divide i and j . Suppose that k and ℓ are not divisible by p . Then we have*

$$|K_{ij}(k, \ell, p)| \leq (i + j)p^{1/2}$$

Proof. The condition on i, j, k, ℓ ensures that $f(z) = kz^i + \ell z^{-j}$ nondegenerate w.r.t. its Newton polyhedron. It is a direct consequence of Thm. 6.1 due to Denef-Loeser. \square

Suppose that f fails to be non degenerate for a given prime p . It happens when p divide either of i, j, k or ℓ . Apriori f can be degenerate in only finitely many cases of prime p . Since we vary p for i, j, k, ℓ fixed, it is trivial to see

$$|K_{ij}(k, \ell, p)| \leq Cp^{1/2}$$

for a constant C independent of p .

If q has many prime factors, we need to reduce the case to the non-degenerate. This will be justified through the next two lemmas.

First we consider the case q being power of a prime p . When k, ℓ divisible by some power of p , first reduction is as follows:

Lemma 6.3. *Let p be a fixed prime. Let $\gcd(k, \ell, p^\alpha) = p^\beta$ and $k' = k/p^\beta, \ell' = \ell/p^\beta$. Then for given positive integers i, j , we have*

$$K_{ij}(k, \ell, p^\alpha) = p^\beta K_{ij}(k', \ell', p^{\alpha-\beta}).$$

Proof. We note that an element $z \in (\mathbb{Z}/p^\alpha \mathbb{Z})^*$ is expressed as

$$z = p^{\alpha-\beta}x + y$$

for $x \in \mathbb{Z}/p^\beta \mathbb{Z}$ and $y \in (\mathbb{Z}/p^{\alpha-\beta} \mathbb{Z})^*$.

Thus, we find that

(22)

$$\begin{aligned} K_{ij}(k, \ell, p^\alpha) &= \sum_{z \in (\mathbb{Z}/p^\alpha \mathbb{Z})^*} \mathfrak{e}\left(\frac{kz^i + \ell z^{-j}}{p^\alpha}\right) = \sum_{x \in \mathbb{Z}/p^\beta \mathbb{Z}} \sum_{y \in (\mathbb{Z}/p^{\alpha-\beta} \mathbb{Z})^*} \mathfrak{e}\left(\frac{k'y^i + \ell'y^{-j}}{p^{\alpha-\beta}}\right) \\ &= p^\beta K_{ij}(k', \ell', p^{\alpha-\beta}). \end{aligned}$$

□

After the previous lemma, we can pull out p -factors out of k, ℓ . For non degenerate m and n , we obtain the following bound:

Lemma 6.4. *Let p be a prime and suppose p does not divide k or ℓ . Then, for given positive integers i, j ,*

$$|K_{ij}(k, \ell, p^\alpha)| \leq (i + j)p^{\frac{\alpha}{2}}.$$

Proof. We are going to use the induction on α . Firstly, the case $\alpha = 1$ is assured by Lemma 6.2.

Following Lemma 12.2-3 of [16], we separately consider the cases even and odd α .

For α even so that $\alpha = 2\beta$, we have

$$(23) \quad K_{ij}(k, \ell, p^{2\beta}) = p^\beta \sum_{\substack{x \in (\mathbb{Z}/p^{2\beta}\mathbb{Z})^* \\ i n x^{i+j} = j m}} \mathfrak{e} \left(\frac{k x^i + \ell x^{-j}}{p^{2\beta}} \right)$$

and for α odd so that $\alpha = 2\beta + 1$,

$$(24) \quad K_{ij}(k, \ell, p^{2\beta+1}) = p^\beta \sum_{\substack{x \in (\mathbb{Z}/p^{2\beta+1}\mathbb{Z})^* \\ i k x^{i+j} = j \ell}} \mathfrak{e} \left(\frac{k x^i + \ell x^{-j}}{p^{2\beta+1}} \right) G_p(x).$$

where

$$G_p(x) = \sum_{y \in \mathbb{Z}/p^\beta \mathbb{Z}} e_p \left(d(x) y^2 + h(x) p^{-\beta} y \right).$$

Here,

$$e_p(x) = \mathfrak{e} \left(\frac{x}{p} \right),$$

$$d(x) = \frac{1}{2} \left(k i (i-1) x^{i-2} + \ell j (j+1) x^{-j-2} \right),$$

$$\text{and } h(x) = k i x^{i-1} - \ell j x^{-j-1}.$$

We have trivial bound for the partial Kloosterman sum

$$\left| \sum_{\substack{x \in (\mathbb{Z}/p^{2\beta}\mathbb{Z})^* \\ i k x^{i+j} = j \ell}} \mathfrak{e} \left(\frac{k x^i + \ell x^{-j}}{p^{2\beta}} \right) \right| \leq i + j$$

and

$$|G_p(x)| \leq p^{1/2}.$$

This completes the proof. □

The above two lemmas imply the Weil bound for $q = p^\alpha$ as follows:

Proposition 6.5. *For all positive integers i, j and positive prime p , $q = p^\alpha$ ($\alpha \geq 1$), we have*

$$|K_{ij}(k, \ell, p^\alpha)| \leq (i+j)(k, \ell, p^\alpha)^{\frac{1}{2}} p^{\frac{\alpha}{2}}.$$

Now it remains to reduce to a single prime factor when there are several prime factors of q .

Lemma 6.6. *For positive integers q_1, q_2 which are relatively prime, we have*

$$K_{ij}(k, \ell, q_1 q_2) = K_{ij}(k \overline{q_1}, \ell \overline{q_1}, q_2) K_{ij}(k \overline{q_2}, \ell \overline{q_2}, q_1),$$

where for $s \neq t \in \{1, 2\}$,

$$\overline{q_s} q_s \equiv 1 \pmod{q_t}.$$

Proof. Notice that the mapping $(\mathbb{Z}/q_1 \mathbb{Z})^* \times (\mathbb{Z}/q_2 \mathbb{Z})^* \rightarrow (\mathbb{Z}/q_1 q_2 \mathbb{Z})^*$ which maps (x, y) to $x q_2 \overline{q_2} + y q_1 \overline{q_1}$ is the isomorphism of the Chinese remainder theorem.

Hence, we can rewrite

$$K_{ij}(k, \ell, q_1 q_2) = \sum_{z \in (\mathbb{Z}/q_1 q_2 \mathbb{Z})^*} \mathfrak{e} \left(\frac{k z^i + \ell z^{-j}}{q_1 q_2} \right)$$

as

$$\begin{aligned} & \sum_{x \in (\mathbb{Z}/q_1 \mathbb{Z})^*} \sum_{y \in (\mathbb{Z}/q_2 \mathbb{Z})^*} \mathfrak{e} \left(\frac{k (x q_2 \overline{q_2} + y q_1 \overline{q_1})^i + \ell (x^{-1} q_2 \overline{q_2} + y^{-1} q_1 \overline{q_1})^j}{q_1 q_2} \right) \\ &= \sum_{x \in (\mathbb{Z}/q_1 \mathbb{Z})^*} \sum_{y \in (\mathbb{Z}/q_2 \mathbb{Z})^*} \mathfrak{e} \left(\frac{k x^i q_2^i \overline{q_2}^i + k y^i q_1^i \overline{q_1}^i + \ell x^{-j} q_2^j \overline{q_2}^j + \ell y^{-j} q_1^j \overline{q_1}^j}{q_1 q_2} \right) \\ &= \sum_{x \in (\mathbb{Z}/q_1 \mathbb{Z})^*} \mathfrak{e} \left(\frac{k x^i q_2^{i-1} \overline{q_2}^i + \ell x^{-j} q_2^{j-1} \overline{q_2}^j}{q_1} \right) \sum_{y \in (\mathbb{Z}/q_2 \mathbb{Z})^*} \mathfrak{e} \left(\frac{k y^i q_1^{i-1} \overline{q_1}^i + \ell y^{-j} q_1^{j-1} \overline{q_1}^j}{q_2} \right) \\ &= K_{ij}(k q_1^{i-1} \overline{q_1}^i, \ell q_1^{j-1} \overline{q_1}^j, q_2) K_{ij}(k q_2^{i-1} \overline{q_2}^i, \ell q_2^{j-1} \overline{q_2}^j, q_1) \\ &= K_{ij}(k \overline{q_1}, \ell \overline{q_1}, q_2) K_{ij}(k \overline{q_2}, \ell \overline{q_2}, q_1). \end{aligned}$$

□

Altogether, we obtain the following Weil type bound for the generalized Kloosterman sums for arbitrary modulus q .

Since $\sum_{q < x} \phi(q) \sim x^2$ for Euler-phi function ϕ , we have the proof of the main theorem from the Weyl's criterion for equi-distribution and forthcoming Cor.6.7.

Combining Prop.6.5 and Lemma 6.6, we have the following result:

Corollary 6.7. *For all positive integers i, j and q ,*

$$|K_{ij}(k, \ell, q)| \leq \left((i+j) \sqrt{(k, \ell)} \right)^{\omega(q)} \sqrt{q},$$

where $\omega(q)$ is the number of prime factors of q .

Proof. Let $q = p_1^{n_1} p_2^{n_2} \dots p_{\omega(q)}^{n_{\omega(q)}}$, for distinct primes $p_1, p_2, \dots, p_{\omega(q)}$.

Since $(k, \ell, p^\alpha) \leq (k, \ell)$, Prop 6.5 directly implies that for any prime p ,

$$|K_{i,j}(k, \ell, p^\alpha)| \leq (i+j) \sqrt{(k, \ell)} \sqrt{p^\alpha}.$$

Moreover, the multiplicity of Kloosterman sum in Lem 6.6, we have above corollary. \square

6.3. Proof of the main theorem. Finally, we deduce the Weyl's criterion from the bound of the generalized Kloosterman sums.

$\omega(q)$ in Cor. 6.7 has well-known estimate:

$$(25) \quad \omega(q) \sim \log \log q.$$

For sufficiently large q ,

$$\left((i+j) \sqrt{(k, \ell)} \right)^{\omega(q)} \leq \left((i+j) \sqrt{(k, \ell)} \right)^{c \log \log q} \leq (\log q)^{c \log(i+j) \sqrt{(k, \ell)}}.$$

Thus, we have that for any $\epsilon > 0$,

$$\left((i+j) \sqrt{(k, \ell)} \right)^{\omega(q)} \ll q^\epsilon.$$

Therefore, we have the following Weil type bound:

Theorem 6.8. *For given pair of positive integers i, j ,*

$$|K_{ij}(k, \ell, q)| \ll q^{\frac{1}{2} + \epsilon}, \quad \forall \epsilon > 0.$$

Now, we show the Weyl's criterion of Generalized dedekind sum from the following estimate:

$$(26) \quad \begin{aligned} & \sum_{0 < q < x} \sum_{\substack{0 < p < q \\ (p, q) = 1}} \mathfrak{e} \left(mR(i, j) q^{N-2} s_{i,j}(p, q) \right) \\ &= \sum_{0 < q < x} K_{i,j} \left(m \alpha_N r_N \binom{N-1}{i}, m \alpha_N r_N \binom{N-1}{j}, q \right) \leq x^{\frac{3}{2} + \epsilon}. \end{aligned}$$

Consequently, Weyl's equidistribution criterion is fulfilled for the fractional part of $mR(i, j) q^{N-2} s_{i,j}(p, q)$:

$$(27) \quad \begin{aligned} & E_{ij}(m, x) = \\ & \frac{1}{\#\{(p, q) | \gcd(p, q) = 1, p < q \leq x\}} \sum_{0 < q < x} \sum_{0 < p < q, (p, q) = 1} \mathfrak{e} \left(mR(i, j) q^{N-2} s_{i,j}(p, q) \right) \rightarrow 0, \end{aligned}$$

as $x \rightarrow \infty$. Therefore the proof of the main theorem is finished.

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E-mail address: byungheup@gmail.com

E-mail address: lee9311@kias.re.kr

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, HOEGIRO 87, DONGDAEMUN-GU, SEOUL 130-722, KOREA